TRINITY COLLEGE

Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section Two: Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	54	35
Section Two: Calculator-assumed	13	13	100	100	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed 65% (100 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

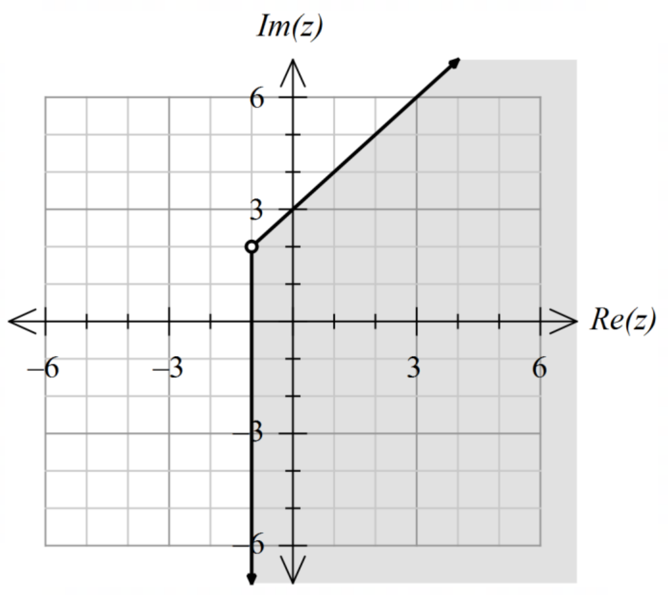
Working time: 100 minutes.

Question 9 **5 marks)**

Solve the equation $z^5 = 16 + 16\sqrt{3}i$, giving solutions in polar form $r\,\text{cis}\,\theta$ where $-\pi < \theta \leq \pi$ and $r > 0$.

Question 10 **(4 marks)**

The locus of a complex number z is shown below.

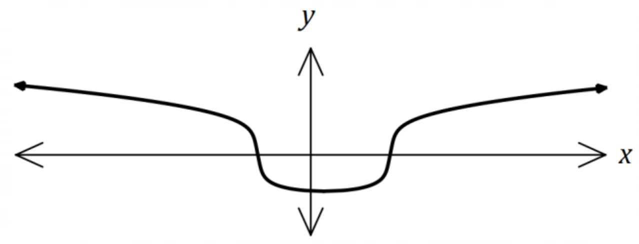


(a) Without using $\text{Re}(z)$ or $\text{Im}(z)$, write an inequality in terms of z for the locus. (3 marks)

(b) Add the locus for $|z| = |z-4|$ to the diagram above. (1 mark)

Question 11 **(6 marks)**

A particle is moving along the curve shown below with equation $x^2 - x = y^5 + y + 6$.

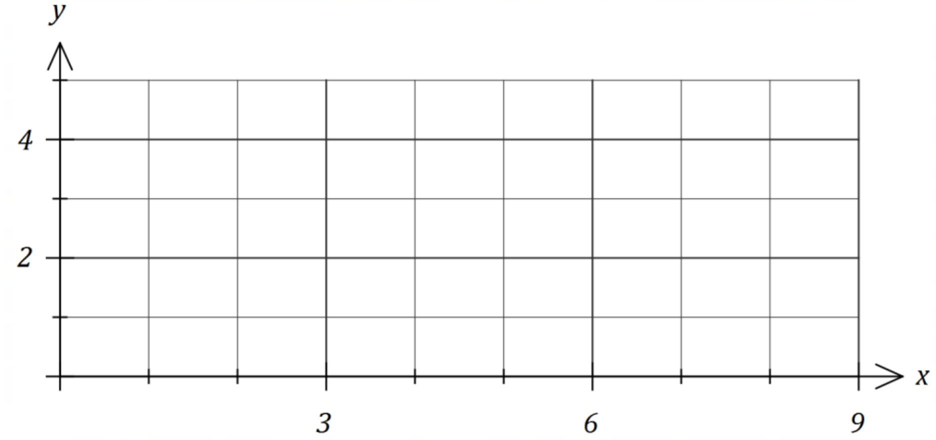


The x-coordinate of the particle is changing at a constant rate given by $\frac{dx}{dt} = -2$.

Determine the rate/s at which the y-coordinate of the particle is changing when $y = 0$.

Question 12 **(5 marks)**

(a) Sketch the graph of $y = \dfrac{e^{0.5x}}{4+x}$ on the axes below. (2 marks)



The Trapezoidal Rule can be used to determine the numerical approximation of a definite integral when an antiderivative cannot be found. When a continuous interval $[a_0,\ a_n]$ is divided into n smaller intervals of equal width w, the bounds of these smaller intervals can be denoted by $a_0,\ a_1,\ a_2, \dots ,\ a_{n-1},\ a_n$. The Trapezoidal Rule is then expressed as follows:

$$\int_{a_0}^{a_n} f(x)\,dx = \frac{w}{2}[f(a_0) + 2f(a_1) + 2f(a_2) + \cdots + 2f(a_{n-1}) + f(a_n)]$$

(b) Use the above rule to determine an estimate to 4 decimal places for $\displaystyle\int_0^8 \frac{e^{0.5x}}{4+x}\,dx$ using 4 intervals. (3 marks)

Question 13 **(8 marks)**

The diameter of copper wire produced by a machine is normally distributed with a mean of $545\ \mu\text{m}$ and a variance of $357\ \mu\text{m}^2$.

A production supervisor routinely takes a random sample of 36 diameters and calculates their mean, $\bar{X}$.

(a) Describe the distribution of $\bar{X}$. (3 marks)

(b) Determine the probability that the mean of a random sample of 36 diameters is less than $540\ \mu\text{m}$. (1 mark)

(c) Repeated random sampling of n diameters from the machine shows that there is a 21% chance that the sample mean exceeds $547\ \mu\text{m}$. Determine n. (4 marks)

Question 14 **(9 marks)**

A virus, that eventually leads to death in just feral cats, is to be released in a large nature reserve where the cats are having a devastating effect on native animals. From previous data, it is expected the rate at which the virus spreads will be modelled by $\frac{dV}{dt} = \frac{V}{10} - \frac{V^2}{800}$ where V is the number of feral cats with the virus and t is the time in days. Five feral cats are initially infected with the virus.

(a) What form of relationship does this differential equation model? (1 marks)

(b) Express V as a function of t in the form $V = \frac{K}{1+Ce^{-at}}$ (4 marks)

(c) Determine the expected number of feral cats to have the virus after two weeks of its release. (1 mark)

(d) For what value/s of V is the growth rate zero? (1 mark)

(e) Determine the time taken to when 75% of the feral cat population are expected have the virus. (2 marks)

Question 15 **(8 marks)**

A researcher used data from a sample of 125 newborn babies in order to estimate the mean weight and length of newborns in a large city.

(a) The weights of the babies in the sample had a mean of 3.28 kg and a standard deviation of 0.62 kg.

(i) Use this data to obtain a 95% confidence interval for the mean weight of a newborn baby in the city. (2 marks)

(ii) State two assumptions made when constructing your confidence interval. (2 marks)

(b) The 99% confidence interval for the mean length L cm of newborn babies derived from the sample was $(49.88, 50.92)$. Determine the sample mean and standard deviation used to construct this interval. (4 marks)

Question 16 **(6 marks)**

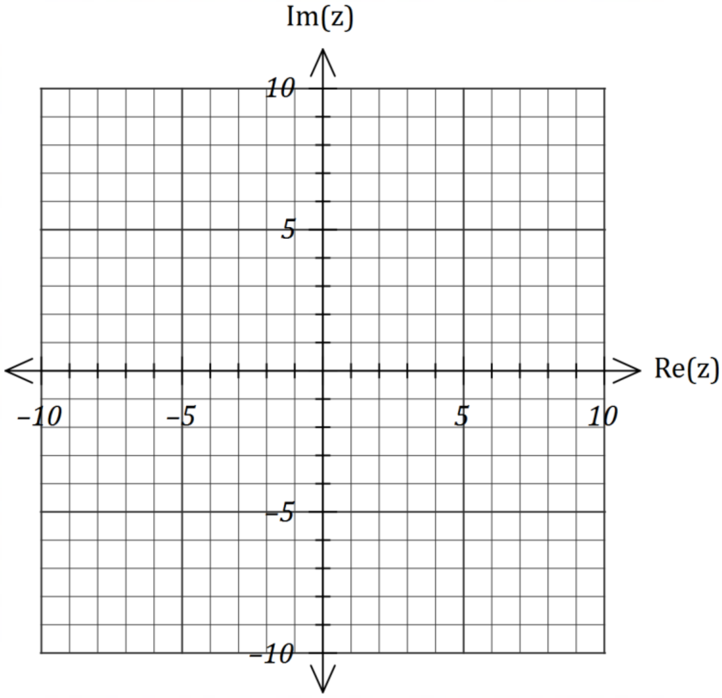
(a) State the equations of all asymptotes of the graph of $y = \dfrac{72 - 5x^2}{x^2 - 36}$. (2 marks)

(b) Let $f(x) = \dfrac{ax^2 + bx + c}{x + d}$.

The graph of $y = f(x)$ has a y-intercept of -9, two asymptotes (with equations $x = -1$ and $y = 5 - 2x$) and no roots. Determine the value of each of the constants a, b, c and d.
(4 marks)

Question 17 **(9 marks)**

(a) Indicate the subset of points in the complex plane that satisfy $|z + 3 - 2i| \leq 4$ on the axes below. (3 marks)



(b) Given that $|z + 3 - 2i| \leq 4$, determine

(i) the maximum value of $\text{Im}(z)$. (2 marks)

(ii) the minimum value of $|z - 2|$. (2 marks)

(iii) the minimum value of $\text{Re}(iz)$. (2 marks)

Question 18 **(12 marks)**

Points A, B and C lie in plane Π and have position vectors $\begin{pmatrix}10\\0\\5\end{pmatrix}$, $\begin{pmatrix}3\\-2\\-3\end{pmatrix}$ and $\begin{pmatrix}5\\-3\\4\end{pmatrix}$ respectively.

Point B also lies on the sphere S that has centre A.

(a) Determine the vector equation of S. (3 marks)

(b) Determine the Cartesian equation for plane Π. (4 marks)

A point and its reflection in a plane are equidistant from the plane and lie on a line that is perpendicular to the plane.

Point P has position vector $\begin{pmatrix} -6 \\ 14 \\ 1 \end{pmatrix}$.

(c) Determine the position vector of P', the reflection of P in plane Π. (5 marks)

Question 19 **(9 marks)**

Researchers used a simulation to model the population of foxes $F(t)$ and the population of rabbits $R(t)$ on an island. The rates of change of each population after t years are given by

$$\frac{dF}{dt} = 0.5R \text{ and } \frac{dR}{dt} = -0.125F.$$

(a) Briefly explain how the rabbit population is changing. (1 mark)

(b) Show that $\dfrac{d^2F}{dt^2} = -0.0625F$. (2 marks)

The equation in part (b) suggests that a model of the form $F(t) = k\sin(at + b)$ would be appropriate, where a, b and k are positive constants.

(c) Explain this choice of model. (1 mark)

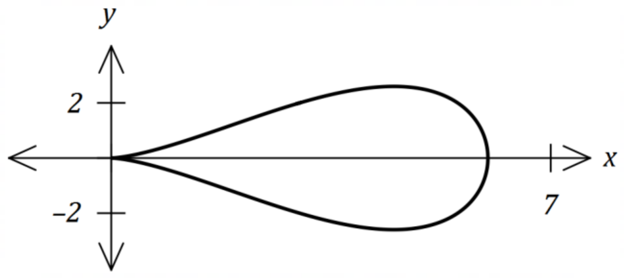
The research model used $b = 0.15$ and the initial size of the fox population was 700.

(d) Determine the value of a and the value of k. (2 marks)

(e) Determine an equation for $R(t)$ in terms of t and use it to calculate the number of years until the rabbit population becomes extinct. (3 marks)

Question 20 **(12 marks)**

The path of a particle is shown in the diagram below.



The position vector of the particle after t seconds is given by $\mathbf{r}(t) = \begin{pmatrix} 3 - 3\sin t \\ 2\cos t - \sin 2t \end{pmatrix}$ centimetres, for $t \geq 0$.

(a) Determine the initial position of the particle. (1 mark)

(b) Determine the acceleration vector of the particle at the instant it first reaches the origin. (4 marks)

(c) Determine the distance travelled by the particle from the time it leaves its initial position until the time it first reaches the origin. (3 marks)

(d) The Cartesian equation of the path of the particle is $ay^2 + bx^3 + 4x^4 = 0$. Determine the value of each of the constants a and b. (4 marks)

Question 21 **(7 marks)**

Water, containing 6 grams of dissolved sugar per litre, flows into a tank at a constant rate of 10 litres per hour.

Water is drawn from the tank, initially containing 250 litres of water with no dissolved sugar, at the same constant rate of 10 litres per hour.

Let the weight of sugar in the tank after t hours be w grams and assume that the sugar is always evenly dissolved throughout the water in the tank.

(a) By considering the rate at which dissolved sugar flows in and out of the tank, show that (2 marks)

$$\frac{dw}{dt} = \frac{1500 - w}{25}.$$

The water drawn from the tank can be used in a manufacturing process once the level of dissolved sugar exceeds 0.75 grams per litre.

(b) Derive an equation for w in terms of t and hence determine how long this will take. (5 marks)

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________

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